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A NECESSARY AND SUFFICIENT CONDITION FOR THE NONEMPTINESS  
OF THE CORES OF PARTITIONING GAMES

Mamoru Kaneko

November 14, 1980

A NECESSARY AND SUFFICIENT CONDITION FOR THE NONEMPTINESS  
OF THE CORES OF PARTITIONING GAMES

by

Mamoru Kaneko<sup>\*</sup>

Abstract: Let  $N$  be a finite set of players and let  $\pi$  be a class of coalitions of  $N$ . We consider games with and without sidepayments such that only the coalitions in  $\pi$  play essential roles but not the others. For an arbitrary  $\pi$ , we get the class of all such games. The purpose of this note is to provide a necessary and sufficient condition with respect to  $\pi$  for the nonemptiness of the cores of all games in the class.

1. Introduction

In situations of  $n$ -person cooperatives games, it is not necessarily equally easy to form every coalition. For example, it might be very hard to form a large coalition because of coalition formation costs. That is, it might often happen that only some coalitions play essential roles but not the others. Myerson [7] considered such a situation from a graph theoretical viewpoint. If a situation, however, has some special structures and even if all coalitions are permitted, it happens that only some coalitions play essential roles, e.g., the assignment game and the assignment market of Shapley and Shubik [10] and Kaneko [4, 5].

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This paper concerns the nonemptiness of the cores of games of such situations. We define games with and without sidepayments called "partitioning games," which are generalizations of the assignment game of [10] and the central assignment game of [5]. The main result of this note is a necessary and sufficient condition for the nonemptiness of the cores of a class of partitioning games.

## 2. The Partitioning Games with and without Sidepayments

Initially we provide partitioning games with sidepayments. Let  $N$  be an arbitrary finite number of players. Let  $N = \{1, 2, \dots, n\}$ . Let  $\pi$  be a class of nonempty coalitions satisfying  $\{i\} \in \pi$  for all  $i \in N$ . We call  $S$  in  $\pi$  a basic coalition. For any nonempty  $S \subset N$ , we call  $p_S = \{T_1, \dots, T_k\}$  a  $\pi$ -partition of  $S$  iff

$$(1) \quad T_t \in \pi \text{ for all } 1, \dots, k \text{ and } p_S \text{ is a partition of } S.$$

Let  $P(S)$  be the set of all  $\pi$ -partitions of  $S$ . We call a game in characteristic function form  $(N, v)$  a partitioning game with sidepayments iff it satisfies

$$(2) \quad v(S) = \max_{p_S \in P(S)} \sum_{T \in p_S} v(T) \text{ for all nonempty } S \subset N.$$

Note that  $v$  satisfies the super-additivity.

The basic idea is very simple. That is, only the basic coalitions can play essential roles in a partitioning game. A typical example is the assignment game of Shapley and Shubik [10].

The core of a game  $(N, v)$  with sidepayments is the set

$$\{x \in R^n : \sum_{i \in N} x_i = v(N) \text{ \& \; } \sum_{i \in S} x_i \geq v(S) \text{ for all nonempty } S \subset N\}, \text{ where}$$

$R^n$  is the  $n$ -dimensional Euclidean space. The following lemma ensures that this definition does not contradict our initial motivation mentioned in Section 1.

Lemma 1. Let  $(N, v)$  be a partitioning game with sidepayments. Then the core coincides with the set  $\{x \in R^n : \sum_{i \in N} x_i = v(N) \text{ \& } \sum_{i \in T} x_i \geq v(T) \text{ for all } T \in \pi\}$ .

Proof. Obvious.

The definition (2) is rewritten in terms of an integer programming as follows: for all nonempty  $S \subset N$ ,

$$(3) \quad v(S) = \max \sum_{\substack{T \in \pi \\ T \subset S}} X_T v(T)$$

$$\text{subject to } \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \text{ for all } i \in S \text{ and } X_T = 0 \text{ or } 1$$

$$\text{for all } T \in \pi \text{ with } T \subset S,$$

where  $(X_T)_{\substack{T \in \pi \\ T \subset S}}$  is a variable. This integer programming is usually called a set partitioning problem. See Balas and Padberg [1] and Murty [6]. This representation will play an important role in considering the nonemptiness of the core of it.

For any  $N$  and  $\pi$ , we denote, by  $GS(N, \pi)$ , the set of all partitioning games with sidepayments which have the set of players  $N$  and the set of basic coalitions  $\pi$ .

Next let us define partitioning games without sidepayments. Let  $N$  and  $\pi$  be given. A partitioning game without sidepayments  $(N, V)$  is a function from the set of all nonempty coalitions to a class of subsets of  $R^n$  such that all nonempty  $S \subset N$ ,

- (4)  $V(S)$  is a closed set in  $R^n$ ,
- (5) if  $x \in V(S)$  and  $y \in R^n$  with  $y_i \leq x_i$  for all  $i \in S$ ,  
then  $y \in V(S)$ ,
- (6)  $\text{Pro}_S[V(S) - \bigcup_{i \in S} \text{interior } V(\{i\})]$  is nonempty and bounded,<sup>1</sup>
- (7)  $V(S) = \bigcup_{p_S \in P(S)} \bigcap_{T \in p_S} V(T)$ .

Assumptions (4)-(6) are the standard technical conditions. Assumption (7) means that when a coalition is formed, the players in  $S$  subdivide  $S$  into a  $\pi$ -partition and get payoffs guaranteed by the basic coalitions. This idea is almost the same as that of partitioning game with sidepayments. A typical example is the central assignment game of Kaneko [5].

The core of a game without sidepayments  $(N, V)$  is the set

$$V(N) = \bigcup_{\substack{S \subset N \\ S \neq \emptyset}} \text{interior } V(S). \text{ Parallel to Lemma 1, the following lemma holds.}$$

Lemma 2. Let  $(N, V)$  be a partitioning game without sidepayments. Then the core coincides with  $V(N) = \bigcup_{S \in \pi} \text{interior } V(S)$ .

Proof. Obvious.

There is, however, a minor conceptual difference between the two above games, i.e., the partitioning game with and without sidepayments. In a partitioning game with sidepayments, it is permitted to transfer money (transferable utility) in every coalition, but in a partitioning game without sidepayments, it is only permitted to transfer something

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<sup>1</sup>  $\text{Pro}_S X = \{(x_i)_{i \in S} : x \in X\}$  for  $S \subset N$  and  $X \subset R^n$ .

in every basic coalition. This difference appears as follows. A game with sidepayments  $(N, v)$  can be represented as a game without sidepayments  $(N, \bar{v})$  such that  $\bar{v}(S) = \{x \in R^n : \sum_{i \in S} x_i \leq v(S)\}$  for all nonempty  $S \subset N$ . Even if  $(N, v)$  is a partitioning game with sidepayments,  $(N, \bar{v})$  is not a partitioning game without sidepayments, i.e., it does not satisfy (7). But as far as we consider the core, any difficulty does not appear. Let us consider another game without sidepayments  $(N, V_v)$  such that

$$(8) \quad V_v(S) = \bigcup_{P_S \in P(S)} \bigcap_{T \in P_S} \bar{v}(S) \quad \text{for all nonempty } S \subset N.$$

Of course,  $(N, V_v)$  is a partitioning game without sidepayments. Then the following lemma holds.

Lemma 3. Let  $(N, v)$  be a partitioning game with sidepayments. Then the core of  $(N, v)$  coincides with the core of  $(N, V_v)$ .

Proof. Obvious.

For any  $N$  and  $\pi$ , we denote, by  $G(N, \pi)$ , the set of all partitioning games without sidepayments which have the set of players  $N$  and the set of basic coalitions  $\pi$ . Embedding  $GS(N, \pi)$  into  $G(N, \pi)$  by the mapping (8):  $v \mapsto V_v$ , we can regard  $GS(N, \pi)$  as a subset of  $G(N, \pi)$ .

We need another concept to state the main result of this note. Let us consider the following system of equations:

$$(9) \quad \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \quad \text{for all } i \in N \quad \text{and} \quad X_T \geq 0 \quad \text{for all } T \in \pi,$$

where  $(X_T)_{T \in \pi}$  is a variable. We say that the system of equations (9) has the integral property iff every extreme solution of (9) consists of integers. If (9) has the integral property, there exists a one-to-one onto mapping from the set of all  $\pi$ -partitions of  $N$  to the set of all extreme solutions of (9). The extreme solutions also coincide with the feasible solutions of (3) in the case of  $S = N$ .

The main result of this note is the following theorem.

Theorem. The following three statements are equivalent:

- (i) The system of equations (9) has the integral property.
- (ii) The core of  $(N, V)$  is nonempty for all  $(N, V)$  in  $G(N, \pi)$ .
- (iii) The core of  $(N, v)$  is nonempty for all  $(N, v)$  in  $GS(N, \pi)$ .

### 3. Remarks

3.1. Let us represent the system of equations (9) as the matrix form, i.e.,  $AX = e$  &  $X \geq 0$ , where  $X = (X_T)_{T \in \pi}$  and  $e$  is the vector every component of which equals 1. A sufficient condition for (9) to have the integral property is the unimodular property of  $A$ , i.e., every minor determinant of  $A$  equals 0, 1 or -1. (Hoffman and Kruskal [2, Theorem 2].) Hoffman and Kruskal gave also several necessary and sufficient conditions and more convenient sufficient conditions for the unimodular property.

Let us consider one necessary and sufficient condition of [2]. For more details, see [2]. Let  $G = (N, E)$  be an oriented graph (i.e.,  $N = \{1, \dots, n\}$  is the set of vertices and  $E$  is the set of edges) (a) which has no circular edges, (b) which has at most one edge between any two given vertices and (c) in which each edge has an orientation.



We call  $(i_1, i_2)$  a direct (inverse) edge iff  $(i_1, i_2) \in E$  ( $(i_2, i_1) \in E$ ). A path is a sequence of distinct vertices  $i_1, i_2, \dots, i_k$ , such that for each  $t$  ( $1 \leq t \leq k-1$ ),  $(i_t, i_{t+1})$  is either a direct or an inverse edge. A path is directed iff every edge is oriented forward. A path is alternating iff successive edges are oppositely oriented. A loop is a path which closes back on itself. A graph is alternating iff every loop in it is alternating.

Let  $\pi = \{T_1, \dots, T_k\}$  be some set of directed path in  $G$ .

Then the incidence matrix  $A = (a_{it})_{\substack{i \in N \\ t=1, \dots, k}}$  is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is in } T_j \\ 0 & \text{otherwise.} \end{cases}$$

Then it holds:

Theorem (Hoffman and Kruskal [2, Theorem 4]): For  $A$  to have a unimodular property, it is sufficient that  $G$  be alternating. If  $\pi$  consists of the set of all directed paths of  $G$ , then for  $A$  to have the unimodular property it is necessary and sufficient that  $G$  be alternating.

It is easily seen that this theorem can be directly applicable to the partitioning games. For example, let us consider the graph drawn in Figure 1 and let  $\pi$  be some set of directed paths with  $\{i\} \in \pi$  for all  $i \in N$ .<sup>2</sup> Then we can construct the classes  $G(N, \pi)$  and  $GS(N, \pi)$ . Since the graph is alternating, the core of every game in  $G(N, \pi)$  is nonempty by the above theorems.

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<sup>2</sup> Every vertex is a path with length 0.

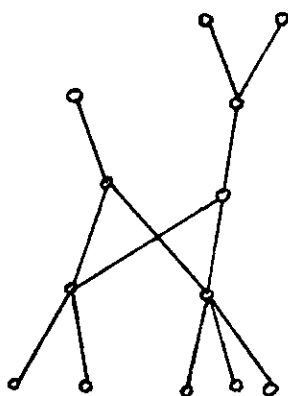


Figure 1. An Alternating Graph (arrows omitted--all should be upward)

3.2. There are a lot of papers on the set partitioning problem. Some of them concerned the integral property of (9) and gave preciser conditions for it. See Balas and Padberg [1].

#### 4. Proof of Theorem

We prove that (i)  $\implies$  (ii)  $\implies$  (iii)  $\implies$  (i). Since  $GS(N, \pi)$  is a subset of  $G(N, \pi)$ , (ii)  $\implies$  (iii) is trivial.

4.1. Proof of (i)  $\implies$  (ii): Let  $(N, V)$  be an arbitrary game in  $G(N, \pi)$ . Due to Scarf's [8] fundamental theorem, it is sufficient to show that  $(N, V)$  is a balanced game under the assumption (i).<sup>3,4</sup>

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<sup>3</sup>Let us call a family  $\gamma$  of nonempty coalitions of  $N$  balanced iff the system of equations

$$\sum_{S: S \ni j} \delta_S = 1 \quad \text{for all } j \in N,$$

has a nonnegative solution with  $\delta_S = 0$  for all  $S \notin \gamma$ . The numbers  $\{\delta_S\}$  are called balanced weights for  $\gamma$ . A game  $(N, V)$  without side-payments is said to be balanced iff the following inclusion statement:

$$\bigcap_{S \in \gamma} V(S) \subset V(N).$$

For any nonempty coalition  $S$ , we define a zero-one matrix

$$A_S = (a_{S:iT})_{\substack{i \in N \\ T \in \pi}} \text{ by}$$

$$(10) \quad \sum_{i \in N} a_{S:iT} = \begin{cases} |T| & \text{or } 0 & \text{if } T \subset S \\ 0 & & \text{if } T \not\subset S \end{cases}$$

$$\sum_{T \in \pi} a_{S:iT} = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

$$a_{S:iT} = 0 \text{ if } i \notin T,$$

where  $|T|$  is the number of members in  $T$ . We call  $A_S$  a partition matrix of  $S$ . We define  $D_S(b) = (d_{S:iT}(b))_{\substack{i \in N \\ T \in \pi}}$  for each  $b \in R^n$  and nonempty coalition  $S$  by

$$(11) \quad d_{S:iT}(b) = \begin{cases} 1 & \text{if } i \in S, i \in T \text{ and } b \in V(T) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that  $p_S$  is a  $\pi$ -partition of  $S$  if and only if  $p_S = \{T : a_{S:iT} = 1 \text{ for some } i \in S\}$  for some partition matrix  $A_S$  of  $S$ . Hence we get the following lemma.

Lemma 4.  $V(S) = \{b \in R^n : D_S(b) \geq A_S \text{ for a partition matrix } A_S \text{ of } S\}$  for all nonempty  $S \subset N$ .

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holds for all balanced families  $\gamma$ . The fundamental theorem of Scarf [8] states that the core of a balanced game with (4), (5) and (6) is nonempty.

<sup>4</sup>This proof is a modification of the proof of the main theorem of Shapley and Scarf [9]. Regrettably, however, their game is not a partitioning game and so, our theorem does not cover their existence theorem.

Let us consider another system of inequalities:

$$(12) \quad \sum_{i \in N} X_{iT} \leq |T| \quad \text{for all } T \in \pi$$

$$\sum_{T \in \pi} X_{iT} = 1 \quad \text{for all } i \in N$$

$$X_{iT} \geq 0 \quad \text{for all } i \in N \text{ \& } T \in \pi, \quad X_{iT} = X_{jT} \quad \text{if} \\ i, j \in T \text{ and } X_{iT} = 0 \quad \text{if } i \notin T,$$

where  $(X_{iT})_{\substack{i \in N \\ T \in \pi}}$  is a variable.

Lemma 5. There exists a one-to-one onto and linear mapping  $f$  from the set of all solutions of (9) to the set of all solutions of (12) such that to any  $(X_T)_{T \in \pi}$  satisfying (9),  $f$  assigns  $(X_{iT})_{\substack{i \in N \\ T \in \pi}}$  such that

$$(13) \quad X_{iT} = \begin{cases} X_T & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is easily verified that the mapping defined by (13) from the set of all solutions of (9) to that of all solutions of (12) is a one-to-one and linear mapping. Conversely we define the mapping which assigns to each  $(X_{iT})_{\substack{i \in N \\ T \in \pi}}$  satisfying (12)  $(X_T)_{T \in \pi}$  such that for all  $T \in \pi$

$$(14) \quad X_T = X_{iT} \quad \text{for some } i \in T.$$

It is easily verified that this mapping is the inverse mapping of the mapping satisfying (13). Q.E.D.

Proof of the Balancedness: Let  $\gamma$  be an arbitrary balanced family of coalitions and let  $b \in \bigcap_{S \in \gamma} V(S)$ . Let  $\{\delta_S\}$  be balanced weights for  $\gamma$ . Then it holds that

$$D_N(b) = \sum_{S \in \gamma} \delta_S D_S(b) .$$

For, if  $d_{N:iT}(b) = 1$ , then  $d_{S:iT}(b) = 1$  if  $i \in S$  and  $d_{S:iT}(b) = 0$  if  $i \notin S$  by (11), which implies  $\sum_{S \in \gamma} \delta_S d_{S:iT}(b) = \sum_{S \in \gamma} \sum_{i \in S} \delta_S d_{S:iT}(b) = 1$ ,

and if  $d_{N:iT}(b) = 0$ , then  $d_{S:iT}(b) = 0$  for all  $S \in \gamma$ , which implies  $\sum_{S \in \gamma} \delta_S d_{S:iT}(b) = 0$ .

By Lemma 4, there is a partition matrix  $A_S$  such that  $D_S(b) \geq A_S$  for all  $S \in \gamma$ , and so we have

$$\sum_{S \in \gamma} \delta_S D_S(b) \geq \sum_{S \in \gamma} \delta_S A_S .$$

Call the matrix on the right  $B$ ; then we have  $D_N(b) \geq B$ . The crucial fact about  $B$  is to satisfy (12). In fact, we have by (10),

$$\begin{aligned} \sum_{i \in N} \sum_{S \in \gamma} \delta_S a_{S:iT} &= \sum_{S \in \gamma} \delta_S \sum_{i \in N} a_{S:iT} = \sum_{S \in \gamma} \delta_S \begin{cases} |T| & \text{or } 0 \text{ if } T \subset S \\ 0 & \text{otherwise} \end{cases} \\ &\leq |T| \sum_{\substack{S \in \gamma \\ S \supset T}} \delta_S \leq |T| ; \\ \sum_{T \in \pi} \sum_{S \in \gamma} \delta_S a_{S:iT} &= \sum_{S \in \gamma} \delta_S \sum_{T \in \pi} a_{S:iT} = \sum_{S \in \gamma} \delta_S \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{\substack{S \in \gamma \\ S \ni i}} \delta_S = 1 . \end{aligned}$$

Since  $a_{S:iT} = 0$  if  $i \notin T$  and it follows from (10) that  $a_{S:iT} = a_{S:jT}$  if  $i, j \in T$ , we get

$$\sum_{S \in Y} \delta_S a_{S:iT} = 0 \text{ if } i \notin T \text{ and}$$

$$\sum_{S \in Y} \delta_S a_{S:iT} = \sum_{S \in Y} \delta_S a_{S:jT} \text{ if } i, j \in T.$$

So,  $B$  satisfies (12). If  $B$  consists of integers, then  $B$  is already a partition matrix of  $N$ , which means  $b \in V(N)$ . Let us assume that  $B$  does not consist of integers. Lemma 4 says that the integral property of (9) is equivalent to that of (12). Since (9) has the integral property by the supposition of proof, (12) has the integral property. Therefore, there are integral extreme solutions of (12) as a convex combination of which  $B$  is represented. Let  $B^1, \dots, B^k$  be such integral extreme solutions. Then  $D_N(b) \geq B^t$  for all  $t = 1, \dots, k$  because  $D_N(b)$  has only zero or one entries. These  $B^t$  are partition matrices. Therefore  $b$  belongs to  $V(N)$ . Q.E.D.

4.2. Proof of (iii)  $\implies$  (i): We assume the negation of (i) and prove that there is a game  $(N, v)$  in  $GS(N, \pi)$  with an empty core.

Since we assume the negation of (i), there is an extreme point  $(x_T^*)_{T \in \pi}$  of the set of all solutions of (9) which is not an integral point. Let  $X$  be the set of all solutions of (9). Since  $X$  is a convex polyhedron, there is a vector  $(u_T)_{T \in \pi}$  such that

$$(15) \quad \sum_{T \in \pi} u_T x_T^* > \sum_{T \in \pi} u_T x_T \text{ for all } (x_T)_{T \in \pi} \in X \text{ with } (x_T)_{T \in \pi} \neq (x_T^*)_{T \in \pi}.$$

Using this  $(u_T)_{T \in \pi}$ , we define a game  $(N, v)$  with sidepayments as follows:

$$\begin{aligned}
 (16) \quad v(S) &= \max_{\substack{T \in \pi \\ T \subset S}} \sum X_T u_T \\
 &\text{subject to } \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \quad \text{for all } i \in S \quad \text{and } X_T = 0 \text{ or } 1 \\
 &\quad \text{for all } T \in \pi \text{ with } T \subset S.
 \end{aligned}$$

By Lemma 1, a necessary and sufficient condition for the nonemptiness of the core of  $(N, v)$  is that  $L \leq v(N)$ , where

$$L = \min_{i \in N} \sum x_i \quad \text{subject to } \sum_{i \in T} x_i \geq v(T) \quad \text{for all } T \in \pi.$$

The dual of the above linear programming is

$$\begin{aligned}
 M &= \max_{T \in \pi} \sum v(T) X_T \\
 &\text{subject to } \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \quad \text{for all } i \in N \quad \text{and } X_T \geq 0 \quad \text{for all } T \in \pi. \quad 5
 \end{aligned}$$

By the duality theorem, it holds that  $L = M$ . Since  $u_T \leq v(T)$  for all  $T \in \pi$ , it holds that  $M' \leq M$ , where

$$\begin{aligned}
 M' &= \max_{T \in \pi} \sum u_T X_T \\
 &\text{subject to } \sum_{\substack{T \in \pi \\ T \ni i}} X_T = 1 \quad \text{for all } i \in N \quad \text{and } X_T \geq 0 \quad \text{for all } T \in \pi.
 \end{aligned}$$

The constraint of the above problem coincides with (9). Hence

$M' = \sum_{T \in \pi} u_T X_T^*$ . Since  $(X_T^*)_{T \in \pi}$  is not a feasible solution of (16) in the case of  $S = N$  because it is not an integral point,  $v(N)$  is smaller than  $M'$  by (15). Hence  $v(N) < M' \leq M = L$ . This implies that the core of  $(N, v)$  is empty. Q.E.D.

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<sup>5</sup>Note that this is a linear programming with mixed constraints. See Goldman and Tucker [3].

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